

Bounds for the Distance Dependence of Correlation Functions of Entangled Photons in Waveguides

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Abstract

The distance dependence of the probability of observing two photons in a waveguide is investigated. The Glauber correlation functions of the entangled photons in waveguides are considered and the spatial and temporal dependence of the correlation functions is evaluated. We derive upper bounds to the distance dependence of the probability of observing two photons. These inequalities should be possible to observe in experiments.

1 Introduction

The transmission of light in waveguides, in particular its quantum properties, is a topic of great interest in optics. Investigations of quantum correlations

and entanglement among photons have been in focus of in the foundations of quantum theory and its applications to quantum information science and metrology.

With the emergence of quantum communication links over long distances [1, 2, 3, 4], there is a need for a detailed study of the dependence of correlations and entanglement among photons on distance.

Since direct interactions between photons in free space are extremely weak, generation of correlated photons generally requires nonlinear media such as the parametric down conversion. Recently, studies of two-photon scattering from a two-level system inside a one-dimensional waveguide have reported various features of photon correlation [5, 6, ?, 7]. In particular a formal scattering theory to study multi-photon transport in a waveguide was employed [8]. We also mention a general model describing nonlinear effects in propagation to and from the system in the quantum state, based on one-dimensional model of field-atom interaction [9], see even [10]. It describes spatiotemporal quantum coherence for the case of spontaneous emission from a single excited atom. In [5] this model was applied to the two-photon input wave packets. This field of research, spatiotemporal behavior of correlations of two photons propagating in nonlinear media, is closely related to studies on nonlinear response of a single atom to an input of two photons, e.g., from a single photon source [11]. This response can be observed in the correlations between the two output photons. Here it is also very important to understand spatiotemporal dependence of correlations.

There are thus sound reasons to study how correlation functions of entangled photons in hollow waveguides behave in space and time. The physical mechanism to model is dispersion that spreads pulses in space and time causing attenuation with distance. The spreading limits also the bit rate for a given waveguide length because of mixing of pulses. Repeaters can be used at some length intervals, which cause higher costs and problems with preserving the quantum state through the repeater. Information on space and time properties of the correlation functions is thus of engineering interest. Such information is provided in the current paper by modelling the effect of modal dispersion.

The study is also a preparation for more elaborate models, including material dispersion in hollow waveguides and fibres, cf. e.g. [12], [13]. To describe the situation, a brief background is presented [14] on optic fibres. For a fibre, it is possible to purify the material to the extent that losses from scattering from the impurities can be neglected in some wavelength bands.

Examples of such bands are the $1.3\ \mu\text{m}$ and $1.55\ \mu\text{m}$ bands. The dispersion can be reduced to a negligible level for a certain wavelength within such a band, and a low dispersion can then be reached for a narrow wavelength band. Such a reduction of dispersion can be reached [14] with a dispersion compensator which is a special fibre with tuned length, after the transmission fibre, having the opposite dispersion to the transmission fibre. Another method [14] is dispersion shift, i.e., a choice of the transversal fibre dimensions. The chromatic dispersion, defined as the combination of modal and material dispersion, can be made to vanish for both methods at the design wavelength.

It is worth mentioning the existence of repeaters for classical optic fibres using optical rather than previously used electrical methods [15] and fibre switches that preserve the quantum state of the photon [16].

We shall study the asymptotic behaviour of the Glauber correlation functions for the entangled states of two photons in waveguides and show their vanishing for large distances. We estimate the rate of decrease of correlations and present upper bounds for the correlation functions.

To estimate the correlation function we shall use some results on the properties of solutions of the (1+1)-dimensional Klein-Gordon equation analogously used in the Haag-Ruelle scattering theory.

We prove that the probability density $P(z_1, t_1, z_2, t_2)$ observing one photon at point z_1 along the waveguide at time t_1 and another photon at point z_2 at time t_2 satisfies the following inequality

$$P(z_1, t_1, z_2, t_2) \leq \frac{C}{(t_0 + |t_1|)(t_0 + |t_2|)} \quad (1)$$

for some t_0 and all z_1, t_1, z_2, t_2 .

The bound (1) and the bounds (26), (27) (see below) should be possible to observe in experiments.

Note that the decreasing of the correlations for entangled states in empty space was found in [?], see also discussion in [?], [?]. In this paper we have considered the waveguides and found the universal bound for the correlations, see also [?] for preliminary considerations.

2 The photon probability density

Let $E_j(\mathbf{r}, t)$ be the j -th component ($j = 1, 2, 3$) of the electric field operator at the space time point \mathbf{r}, t . The operator can be written as the sum of the

positive and negative frequency parts: $E_j(\mathbf{r}, t) = E_j^{(+)}(\mathbf{r}, t) + E_j^{(-)}(\mathbf{r}, t)$. The probabilities of photo detection are given by Glauber's formulas, [17]. In particular, the probability that a state ψ of the radiation field will lead to the detection at time t of a photon with the polarization along the direction j by a detector atom placed at point \mathbf{r} is proportional to the first-order correlation function $P_\psi(\mathbf{r}, t, j) = \langle \psi | E_j^{(-)}(\mathbf{r}, t) E_j^{(+)}(\mathbf{r}, t) | \psi \rangle$. The joint probability of observing one photo ionization with polarization j_1 at point \mathbf{r}_1 at time between t_1 and $t_1 + dt_1$ and another one with polarization j_2 at point \mathbf{r}_2 between t_2 and $t_2 + dt_2$ with $t_1 \leq t_2$ is proportional to $P_\psi(\mathbf{r}_1, t_1, j_1; \mathbf{r}_2, t_2, j_2) dt_1 dt_2$, where the second-order correlation function is

$$P_\psi(\mathbf{r}_1, t_1, j_1; \mathbf{r}_2, t_2, j_2) = \langle \psi | E_{j_1}^{(-)}(\mathbf{r}_1, t_1) E_{j_2}^{(-)}(\mathbf{r}_2, t_2) E_{j_2}^{(+)}(\mathbf{r}_2, t_2) E_{j_1}^{(+)}(\mathbf{r}_1, t_1) | \psi \rangle \quad (2)$$

The waveguide is a hollow conducting cylindrical tube $\Gamma \subset \mathbb{R}^3$ along the z axis with boundary surface S and with a cross section Ω with the bounding curve $\partial\Omega$ in the xy -plane. It will be assumed that the walls have infinite conductivity. Appropriate boundary conditions are posed: $E_t|_S = 0$, $H_n|_S = 0$, where E_t is the component of electric field \mathbf{E} tangential to the boundary of the waveguide and H_n is the component of magnetic field \mathbf{H} normal to the boundary.

It is well known that in the interior of the waveguide the solutions of the Maxwell equations without sources can be divided into two sets of solutions, the so called *TM* modes with $H_z = 0$ and the *TE* modes with $E_z = 0$ [19, 20]. A general solution is a linear combination of the *TM* and *TE* modes.

The general solution of the Maxwell equations in the waveguide can be written as follows [18]. Let $\varphi_{n\nu}(z, t)$ be any function satisfying the Klein-Gordon equation

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial z^2} + m_{n\nu}^2 \right) \varphi_{n\nu}(z, t) = 0. \quad (3)$$

Here $n = 1, 2, \dots$ and $\nu = TM$ or TE . We define for $\nu = TM$

$$\begin{aligned} \mathbf{E}_{n\nu}(\mathbf{r}, t) &= \mathbf{e}_{n\nu}(x, y) m_{n\nu}^{-1} \frac{\partial}{\partial z} \varphi_{n\nu}(z, t) \\ &\quad + \mathbf{n}_z m_{n\nu} v_n(x, y) \varphi_{n\nu}(z, t), \\ \mathbf{H}_{n\nu}(\mathbf{r}, t) &= -\mathbf{h}_{n\nu}(x, y) m_{n\nu}^{-1} \frac{\partial}{\partial t} \varphi_{n\nu}(z, t) \end{aligned} \quad (4)$$

where v_n is the solution of the eigenvalue problem

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + m_n^2\right)v_n(x, y) = 0, \quad (x, y) \in \Omega, \quad (5)$$

$$v_n|_{\partial\Omega} = 0$$

with the properties

$$\int_{\Omega} v_n(x, y)v_{n'}(x, y)dxdy = \delta_{nn'}, \quad (6)$$

$$\sum_n v_n(x, y)v_n(x', y') = \delta(x - x')\delta(y - y'), \quad (7)$$

where $m_n^2 > 0$ are the eigenvalues.

It is defined for $\nu = TM$

$$\mathbf{e}_{n\nu}(x, y) = \nabla_T v_n(x, y), \quad (8)$$

$$\mathbf{h}_{n\nu}(x, y) = \mathbf{n}_z \times \nabla_T v_n(x, y) = \mathbf{n}_z \times \mathbf{e}_{n\nu}(x, y).$$

$\mathbf{E}_{n\nu}(\mathbf{r}, t)$ and $\mathbf{H}_{n\nu}(\mathbf{r}, t)$ are defined in an analogous manner [?] for $\nu = TE$.

The general solution of the Maxwell equations in the waveguide can now be written in the form

$$\mathbf{E}(\mathbf{r}, t) = \sum_{n\nu} \mathbf{E}_{n\nu}(\mathbf{r}, t), \mathbf{H}(\mathbf{r}, t) = \sqrt{\frac{\epsilon_0}{\mu_0}} \sum_{n\nu} \mathbf{H}_{n\nu}(\mathbf{r}, t). \quad (9)$$

We write the solution of the Klein-Gordon Eq. (3) in the form

$$\begin{aligned} \varphi_n(z, t) = & \int \frac{dk}{2\sqrt{2\pi\omega_n(k)}} (a_n^+(k)e^{i\omega_n(k)t - ikz} \\ & + a_n(k)e^{-i\omega_n(k)t + ikz}), \end{aligned} \quad (10)$$

where $\omega_n(k) = \sqrt{k^2 + m_n^2}$, and quantize it by taking $a_n(k), a_n^+(k)$ as the annihilation and creation operators.

Now the quantum electromagnetic field in the waveguide is reduced to a set of massive (1+1)-dimensional Klein-Gordon fields. Let us consider one of the modes. We define a one particle state

$$|\psi_1\rangle = \int g(k)a_k^\dagger|0\rangle, \quad (11)$$

where $|0\rangle$ is the Fock vacuum. The probability density to detect the photon at the point z along the waveguide at time t is proportional to

$$P(z, t) = \langle \psi_1 | \varphi^{(-)}(z, t) \varphi^{(+)}(z, t) | \psi_1 \rangle, \quad (12)$$

where

$$\varphi^{(+)}(z, t) = \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} \frac{dk}{\sqrt{2\omega_k}} a_k e^{-i\omega_k t + ikz}, \quad \varphi^{(-)}(z, t) = h.c., \quad (13)$$

and $\omega_k = \sqrt{k^2 + m^2}$, $m > 0$.

The expression (12) can be written as

$$P(z, t) = |A(z, t)|^2, \quad (14)$$

where

$$A(z, t) = \langle 0 | \varphi^{(+)}(z, t) | \psi_1 \rangle = \int dk g(k) \frac{e^{-i\omega_k t + ikz}}{2\sqrt{2\pi\omega_k}}. \quad (15)$$

The function $A(z, t)$ is a solution of the Klein-Gordon equation. Let us consider the question of how the solution behaves in the limit of large t in a frame of reference moving with a constant velocity $v \leq V < 1$. In this frame, corresponding to the time $x(1 - v)/v$ after the wave front at $t = x$, the field is given by

$$A(vt, t) = \int dk g(k) \frac{e^{-it(\omega_k - kv)}}{2\sqrt{2\pi\omega_k}}. \quad (16)$$

For sufficiently smooth function $g(k)$ one can use the stationary phase method to get

$$A(vt, t) = g(k_0) \frac{e^{-it(\omega_{k_0} - k_0 v) - i\pi/4}}{2\sqrt{2\pi\omega_{k_0}}} \sqrt{\frac{2\pi}{t\omega''(k_0)}} + O\left(\frac{1}{t}\right). \quad (17)$$

Here k_0 is the solution of the equation $\omega'(k) = v$, i.e. $k_0 = mv/\sqrt{1 - v^2}$ and one has $\omega''(k_0) > 0$. For the probability $P(z, t)$ we obtain

$$P(vt, t) = \frac{1}{t} \frac{|g(k_0)|^2}{4\omega_{k_0}\omega''(k_0)} + O\left(\frac{1}{t^{3/2}}\right). \quad (18)$$

More elaborated results on asymptotic expansions of the solution to the Klein-Gordon equation are given by Hörmander [22].

3 The entangled photon correlation function with distance dependence

Now we define a two particle entangled state (biphoton)

$$|\psi\rangle = \int f(k_1, k_2) a_{k_1}^\dagger a_{k_2}^\dagger |0\rangle, \quad (19)$$

where $|0\rangle$ is the Fock vacuum and $f(k_1, k_2)$ is the two-photon wave function which is a symmetric function, $f(k_1, k_2) = f(k_2, k_1)$ because we deal with bosons.

The probability to detect one particle at the point z_1 along the waveguide at time t_1 and another particle at the space point z_2 at time t_2 is proportional to

$$P(z_1, t_1, z_2, t_2) = \langle \psi | \varphi^{(-)}(z_1, t_1) \varphi^{(-)}(z_2, t_2) \varphi^{(+)}(z_2, t_2) \varphi^{(+)}(z_1, t_1) | \psi \rangle. \quad (20)$$

The expression for $P(z_1, t_1, z_2, t_2)$ (20) can be written as

$$P(z_1, t_1, z_2, t_2) = |A(z_1, t_1, z_2, t_2)|^2, \quad (21)$$

where

$$A(z_1, t_1, z_2, t_2) = \langle 0 | \varphi^{(+)}(z_1, t_1) \varphi^{(+)}(z_2, t_2) | \psi \rangle = \int dk_1 dk_2 \quad (22)$$

$$\left\{ \frac{e^{-i\omega_{k_2} t_2 + ik_2 z_2}}{2\sqrt{2\pi\omega_{k_2}}} \frac{e^{-i\omega_{k_1} t_1 + ik_1 z_1}}{2\sqrt{2\pi\omega_{k_1}}} f(k_1, k_2) + (k_1 \leftrightarrow k_2) \right\}.$$

Note that the function $A(z_1, t_1, z_2, t_2)$ satisfies the Klein-Gordon equation with respect to z_1, t_1 and z_2, t_2 .

Let us suppose that one of the photons is observed in a frame of reference moving with velocity v_1 and another photon is observed in a frame of reference moving with velocity v_2 . By using the stationary phase method we obtain for large t_1 and t_2 :

$$A(z_1, t_1, z_2, t_2) = \sqrt{\frac{2\pi}{t_1 \omega''(k_{10})}} \sqrt{\frac{2\pi}{t_2 \omega''(k_{20})}} f(k_{10}, k_{20}) \frac{1}{2\sqrt{2\pi\omega_{k_{10}}}} \frac{1}{2\sqrt{2\pi\omega_{k_{20}}}} e^{-i\pi/2} \quad (23)$$

$$\{e^{-it_1(\omega_{k_{10}} - ik_{10}v_1)} e^{-it_2(\omega_{k_{20}} - ik_{20}v_2)} + (k_1 \leftrightarrow k_2)\}.$$

Here $k_{10} = mv_1/\sqrt{1-v_1^2}$, $k_{20} = mv_2/\sqrt{1-v_2^2}$, $z_1 = v_1 t_1$ and $z_2 = v_2 t_2$.

Therefore

$$P(z_1, t_1, z_2, t_2) = \frac{|f(k_{10}, k_{20})|^2}{16t_1 t_2 \omega''(k_{10}) \omega''(k_{20}) \omega_{k_{10}} \omega_{k_{20}}}. \quad (24)$$

$$|\{e^{-it_1(\omega_{k_{10}} - ik_{10}v_1)} e^{-it_2(\omega_{k_{20}} - ik_{20}v_2)} + (k_1 \leftrightarrow k_2)\}|^2$$

It is interesting to see the difference between the entangled wave function $f(k_{10}, k_{20})$ and the separable one by looking to it with an explicitly indicated dependence on the spacetime coordinates:

$$f(k_{10}, k_{20}) = f\left(m\frac{z_1}{t_1}/\sqrt{1-\frac{z_1^2}{t_1^2}}, m\frac{z_2}{t_2}/\sqrt{1-\frac{z_2^2}{t_2^2}}\right) \quad (25)$$

Let the wave function of two photons $f(k_1, k_2)$ in a waveguide be a smooth fast decreasing function. Then the probability of observing two photons in the waveguide should satisfy the following bounds. For any $n_1, n_2 = 0, 1, 2, 3, \dots$ there exist constants $C_{n_1 n_2}$ such that for

$$|z_1| \geq |t_1|, |z_2| \geq |t_2|, \quad (26)$$

one has

$$P(z_1, t_1, z_2, t_2) \leq \frac{C_{n_1 n_2}}{(1 + |z_1|)^{n_1} (1 + |z_2|)^{n_2}}. \quad (27)$$

Furthermore there exists a constant C such that

$$P(z_1, t_1, z_2, t_2) \leq \frac{C}{|t_1| |t_2|}, \quad (28)$$

for all z_1, t_1, z_2, t_2 . An asymptotic estimate for C , valid for large t_1 or t_2 , is provided by (24).

An explicit expression for the wave function of the biphotons in a special case is given in [21]:

$$f(k_1, k_2) = \frac{i}{k_0^2} f_P(k_1 + k_2) \sqrt{6k_1 k_2 (k_1 + k_2)}, \quad (29)$$

where $f_P(k_1 + k_2)$ is a Gaussian function describing the pumping photons. By using this form of the wave function we obtain the asymptotic formula for the probability in this special case.

To conclude, the main result of this paper is the bounds to the probability density (1) and (26), (27) which, in principle, should be possible to test in experiments.

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